

Semicontinuity of 4d $\mathcal{N} = 2$ spectrum under renormalization group flow

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ABSTRACT: We study renormalization group flow of four dimensional $\mathcal{N} = 2$ SCFTs defined by isolated hypersurface three-fold singularities. We define the spectrum of $\mathcal{N} = 2$ theory as the set of scaling dimensions of the parameters on the Coulomb branch, which include Coulomb branch moduli, mass parameters and coupling constants. We prove that the spectrum of those theories is semicontinuous under the RG flow on the Coulomb branch using the mathematical result about the singularity spectra under deformation. The semicontinuity behavior of $\mathcal{N} = 2$ spectrum implies a theorem under relevant and Coulomb branch moduli deformation, the absence of dangerous irrelevant deformations and can be taken as the necessary condition for the ending point of a RG flow. This behavior is also true for (c, c) ring deformation of two dimensional Landau-Ginzburg model with $(2, 2)$ supersymmetry.

KEYWORDS: Renormalization Group, Supersymmetry and Duality

ARXIV EPRINT: [1510.06036](https://arxiv.org/abs/1510.06036)

Contents

1	Introduction	1
2	Spectrum of $\mathcal{N} = 2$ SCFT	3
3	Spectra of singularity and $\mathcal{N} = 2$ SCFT	4
4	Semicontinuity of spectrum under renormalization group flow	6
4.1	Consequences	7
4.2	Semicontinuity of $\mathcal{N} = 2$ spectrum	7
4.3	A theorem	7
4.4	Absence of dangerous irrelevant deformations	8
4.5	Necessary condition for the ending point of a RG flow	8
5	Conclusion	8

1 Introduction

The understanding of renormalization group (RG) flow between two conformal field theories is a central subject in study of quantum field theory [1]. In particular, we would like to know how quantities associated with conformal field theory is changed under RG flow. The most well studied quantity is the central charge a in even dimension, and one expect the following a theorem:

$$a_{\text{UV}} > a_{\text{IR}}. \quad (1.1)$$

This theorem has been studied for 2d theory in [2], for 4d theory [3–6], and for 6d supersymmetric field theory in [7].

A natural further question is that can we say something about other physical quantities along the RG flow? Gukov is trying to study how the number of relevant operators is changed under RG flow motivated by Morse theory [8], and he proposed a so-called μ theorem. There are several directions that one could extend such line of study, for example one might want to study how the scaling dimension of an operator changes under RG flow and one should also include operators whose expectation value parameterizes the moduli space of vacua into the study.

In this paper, we would like to study those RG flow questions for four dimensional $\mathcal{N} = 2$ SCFTs. The Coulomb branch of $\mathcal{N} = 2$ theory has a remarkable Seiberg-Witten (SW) solution [9, 10], and we can determine the IR phase exactly. One can also get various interesting SCFTs by turning on relevant operators and the expectation values of Coulomb branch operators. Therefore $\mathcal{N} = 2$ theory is a natural place to study RG flows.

Given the importance of Coulomb branch of a $\mathcal{N} = 2$ theory, we define the spectrum of a $\mathcal{N} = 2$ theory as the set of scaling dimensions of parameters on the Coulomb branch. Those parameters include the Coulomb branch moduli (expectation value of $\mathcal{N} = 2$ chiral primary operators), coupling constants, and masses. Let's denote the rank of charge lattice of a theory \mathcal{T} as μ , then $\mathcal{N} = 2$ spectrum consists a set of ordered rational numbers:

$$(\Delta_1, \Delta_2, \dots, \Delta_\mu), \quad (1.2)$$

and here we have $\Delta_1 \geq \Delta_2 \dots \geq \Delta_\mu$.

Let's focus on the theory defined by isolated three-fold hypersurface singularity [11, 12]. The SW solutions of this class of theories are given by the mini-versal deformation of the singularity, so the RG flow and its ending point is captured by the deformation of the singularity. $\mathcal{N} = 2$ spectrum can be computed from the spectra of singularity, which is a set of μ rational numbers $sp(f) = (l_1, l_2, \dots, l_\mu)$ with ordering $l_1 \leq l_2 \dots \leq l_\mu$. The $\mathcal{N} = 2$ spectrum can be found using the following simple formula:

$$\Delta_\alpha = 1 + \frac{1 - l_\alpha}{l_1}. \quad (1.3)$$

For each subset B on the real line, we can define a non-negative integer counting the number of spectra falling in B :

$$|sp_B(f)| = sp(f) \cap B; \quad (1.4)$$

A subset B is called a semicontinuity domain if

$$|sp_B(f_{UV})| \geq |sp_B(f_{IR})|, \quad (1.5)$$

for any deformation of f_{UV} . The spectrum has the following amazing semicontinuity property which is first conjectured by Arnold [13], and proven in various special cases in [14–16]. The relevant theorem for us is the one proven by Steenbrink [17]:

Theorem 1. Every half open interval $(t, t+1]$ is a semicontinuity domain of deformations of isolated hypersurface singularity.

This theorem has various interesting physical consequences. A first consequence is about how the spectrum is changed under RG flow. Let's assume the ending point of RG flow has the spectrum $(\Delta_{1'}, \Delta_{2'}, \dots, \Delta_{\mu'})$, then using above theorem we can show that every RG flow satisfy the following conditions

$$\Delta_1 \geq \Delta_{1'}, \quad \Delta_2 \geq \Delta_{2'}, \dots; \quad (1.6)$$

This is a much stronger result than a theorem, and we can actually use it to prove a theorem. Using above theorem, we can prove the absence of dangerous irrelevant deformations, and also put constraints on the possible ending point of a RG flow.

This paper is organized as follows: section 2 gives a detailed discussion about the spectrum of $\mathcal{N} = 2$ SCFT; section 3 discusses the identification of $\mathcal{N} = 2$ spectrum and the spectrum of singularity; section 4 uses the theorem on change of singularity spectra under deformation to prove semicontinuity of $\mathcal{N} = 2$ spectrum. Finally, a conclusion is given in section 5.

2 Spectrum of $\mathcal{N} = 2$ SCFT

Four dimensional $\mathcal{N} = 2$ SCFT has a $SU(2)_R \times U(1)_R$ R symmetry. The representation of $\mathcal{N} = 2$ superconformal algebra has been studied in complete detail in [18–20]. A generic representation is labeled as $|\Delta, R, r, j, \bar{j}\rangle$, here Δ is the scaling dimension, R is $SU(2)$ representation, r is $U(1)$ charge, and (j, \bar{j}) are left and right spin. There are two sets of special short representations which is of crucial importance for us: $\mathcal{E}_{r,(0,0)}$ and $\hat{\mathcal{B}}_R$ (we use the notation of [18]). The scaling dimension of an operator $\mathcal{E}_{r,(0,0)}$ is $\Delta = \frac{1}{2}r$ (the scaling dimension of lowest component of the supermultiplet), and the scaling dimension of $\hat{\mathcal{B}}_R$ is $\Delta = 2R$.

$\mathcal{E}_{r,(0,0)}$ can be separated into three different sets: a: relevant operators if $1 < \Delta < 2$; b: exact marginal operator if $\Delta = 2$; c: others if $\Delta > 2$. For the relevant operators and exact marginal operators, one can add the following deformations to our theory

$$\delta S = \lambda \int d^4\theta \mathcal{E}_{r,(0,0)} + c.c.; \quad (2.1)$$

We can assign a scaling dimension to λ so that the total deformation is a marginal deformation:

$$\Delta(\lambda) + \Delta(\mathcal{E}_{r,(0,0)}) = 2; \quad (2.2)$$

One can form a different type of relevant deformation using $\hat{\mathcal{B}}_R$:

$$\delta S = m \int d^2\theta \hat{\mathcal{B}}_1 + c.c.; \quad (2.3)$$

$\hat{\mathcal{B}}_1$ has scaling dimension 2, and m has scaling dimension one. The above two deformations are the only relevant or exact marginal deformations for $\mathcal{N} = 2$ SCFT, see [20].

The Coulomb branch are parameterized by the expectation value of the chiral primary operators $\mathcal{E}_{r,(0,0)}$, and it is conjectured that the chiral ring of those operators are a free generated finite ring [19]. The low energy effective field theory is described by Seiberg-Witten solution which depends on the coupling constants λ and masses m , and expectation value u of chiral operators $\mathcal{E}_{r,(0,0)}$. So the Seiberg-Witten solution has the following schematic form:

$$F(z, u, \lambda, m) = 0; \quad (2.4)$$

Using SW solution, we can determine the IR phase for a fixed UV parameters (u, λ, m) . The Coulomb branch plays a crucial role in studying $\mathcal{N} = 2$ theory, i.e. the central charge can be purely computed from the structure of Coulomb branch.

Let's use μ to denote the rank of charge lattice $\mu = 2r + f$, with r the number of Coulomb branch moduli (number of independent $\mathcal{E}_{r,(0,0)}$ operators in Coulomb branch chiral ring), and f the number of mass parameters. We define $\mathcal{N} = 2$ spectrum as a set of μ ordered rational numbers:

$$(\Delta_1, \Delta_2, \dots, \Delta_\mu); \quad (2.5)$$

with ordering $\Delta_1 \geq \Delta_2 \dots \geq \Delta_\mu$. Here Δ_i the scaling dimension of the operators $\mathcal{E}_{r,(0,0)}$ and coupling constants λ_i, m . For operators $\mathcal{E}_{r,(0,0)}$ with scaling dimension larger than two,

we add negative numbers $\Delta[\lambda]$ such that $\Delta(\lambda) + \Delta[\mathcal{E}_{r,(0,0)}] = 2$, notice that such added λ parameters would not change the low energy effective theory, and they can be thought of irrelevant deformation. The spectrum has the following obvious properties:

- The spectrum is symmetric with respect to number 1. This is due to definition.
- The spectrum does not depend on the exact marginal deformation, and this can be seen from the representation theory of $\mathcal{N} = 2$ superconformal algebra [18], i.e. the operators $\mathcal{E}_{r,(0,0)}$ and $\hat{\mathcal{B}}_R$ could not combine with other operators to form long multiplets. Obviously the spectrum does not depend on irrelevant deformations.
- The spectrum satisfies the following condition

$$\sum \Delta_i = \mu. \quad (2.6)$$

Example 1. Let's consider A_3 Argyres-Douglas theory [21, 22], and its Seiberg-Witten curve is $x^2 = z^4 + u_1 z^2 + u_2 z + u_3$. The charge lattice has rank three. The scaling dimensions of various parameters are $[u_1] = \frac{2}{3}, [u_2] = 1, [u_3] = \frac{4}{3}$, so the spectrum of this theory is

$$\left(\frac{4}{3}, 1, \frac{2}{3}\right). \quad (2.7)$$

Example 2. Let's consider T_3 theory [23, 24]. This theory has six mass parameters and one dimension 3 operators, so the spectrum is

$$(3, 1, 1, 1, 1, 1, -1). \quad (2.8)$$

The Higgs branch is parameterized by the expectation value of operators $\hat{\mathcal{B}}_R$, but now the chiral ring is not free generated. However, we would like to conjecture that the chiral ring of these operators is a finite generated graded affine ring:

$$\mathbb{C}[x_1, \dots, x_n]/\mathbb{I}. \quad (2.9)$$

It would be nice to have a way to compute this affine ring for general $\mathcal{N} = 2$ theory, see [25, 26] for some examples. More generally, one can have mixed branch which is a direct sum of a Coulomb component and a Higgs component, see [27] for the structure of moduli space of class \mathcal{S} theory.

One can have various interacting SCFTs on Coulomb branch, and it is a rather interesting and difficult problem to classify the IR SCFTs which can appear on the Coulomb branch of a given UV SCFT. The IR theory on the Higgs branch is rather simple, which is simply a bunch of free hypermultiplets. So all the interesting RG flow happens on the Coulomb branch for $\mathcal{N} = 2$ theory.

3 Spectra of singularity and $\mathcal{N} = 2$ SCFT

A large class of $\mathcal{N} = 2$ SCFT can be defined by a three dimensional isolated quasi-homogeneous hypersurface singularity $f : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}, 0)$ [11, 12]. The \mathbb{C}^* action defined on f is

$$f(\lambda^{q_i} z_i) = \lambda f(z_i); \quad (3.1)$$

To define a 4d SCFT, we require $\sum q_i > 1$. The SW geometry for the Coulomb branch can be described by the mini-versal deformation of the singularity:

$$F(z, \lambda) = f(z_0, z_1, z_2, z_3) + \sum_{\alpha=1}^{\mu} \lambda_{\alpha} \phi^{\alpha} = 0; \quad (3.2)$$

Here ϕ_{α} is the monomial basis of the Jacobi algebra:

$$J_f = \mathbb{C}[z_0, z_1, z_2, z_3] / \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \right); \quad (3.3)$$

and its dimension is denoted as μ . The scaling dimension of parameter λ_{α} is

$$\Delta_{\alpha} = [\lambda_{\alpha}] = \frac{1 - Q_{\alpha}}{\sum_{i=0}^3 q_i - 1}; \quad (3.4)$$

Here Q_{α} is the charge of the monomial ϕ_{α} under the \mathbb{C}^* action, and those scaling dimensions are exactly the spectrum of the corresponding $\mathcal{N} = 2$ theory.

Using mixed hodge structure [28], one can define a set of μ rational numbers for the isolated singularity:

$$(l_1, l_2, \dots, l_{\mu}), \quad (3.5)$$

and the ordering is $l_1 \leq l_2, \dots \leq l_{\mu}$ (These numbers are also related to eigenvalues of the monodromy group associated with the singularity: $m_i = \exp(2\pi i l_i)$), and they are called singularity spectra. For quasi-homogeneous isolated singularity, the spectrum can be computed from the Jacobi algebra J_f . For a monomial basis $\phi_{\alpha} = z_0^{n_0} z_1^{n_1} z_2^{n_2} z_3^{n_3}$, one associate a spectrum number:

$$l_{\alpha} = \sum_{i=0}^3 n_i q_i + \sum_{i=0}^3 q_i - 1; \quad (3.6)$$

The lowest number in the spectrum of singularity is $l_1 = \sum_{i=0}^3 q_i - 1$ and the maximal number is $l_{\mu} = 4 - 2 \sum_{i=0}^3 q_i + l_1$. For the singularity defining a 4d $\mathcal{N} = 2$ theory, the spectra is in the range $(0, 2)$.

It is easy to find out $\mathcal{N} = 2$ spectrum from singularity spectrum using the following formula:

$$\Delta_{\alpha} = 1 + \frac{1 - l_{\alpha}}{l_1}; \quad (3.7)$$

It is interesting to note that $l_{\alpha} = 1$ gives us a mass parameter: $l_{\alpha} = 1 \rightarrow \Delta_{\alpha} = 1$.

For general isolated three-fold hypersurface singularity, the spectrum of singularity has the following properties [29]:

- It is symmetric with respect to 1.
- The sum of the spectrum is $\sum_{\alpha=1}^{\mu} l_{\alpha} = \mu$, which implies $\sum_{\alpha=1}^{\mu} \Delta_{\alpha} = \mu$.

The second property is equivalent to $\sum_{\alpha=1}^{\mu} \Delta_{\alpha} = \mu$. These two properties agree exactly with those of the $\mathcal{N} = 2$ spectrum.

Example. Let's consider A_3 AD theory defined by the singularity $f = z_0^2 + z_1^2 + z_2^2 + z_3^4$. The weights are $(q_0, q_1, q_2, q_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4})$. The basis of Jacobi algebra is $(1, z_3, z_3^2)$, so the spectra of the singularity is $(\frac{3}{4}, 1, \frac{5}{4})$ using formula (3.6).

Remark. In previous work, people usually studied isolated singularity with a C^* action to define a 4d $\mathcal{N} = 2$ SCFT [11, 12]. Here we want to conjecture that a general isolated three-fold singularity should also define a $\mathcal{N} = 2$ SCFT (The C^* action might be thought of as accidental symmetry for the Landau-Ginzburg model on worldsheet if we use type IIB string description [30, 31]). The $\mathcal{N} = 2$ spectrum is computed from singularity spectrum using formula (3.7), and the necessary condition for the existence of a SCFT is $l_1 > 0$ (This is the condition so that the singularity is rational.).

4 Semicontinuity of spectrum under renormalization group flow

The Coulomb branch is parameterized by the complex numbers λ_α , and λ_α can be classified by their scaling dimensions:

- $\Delta_\alpha > 1$: Coulomb branch moduli.
- $0 < \Delta_\alpha < 1$: relevant deformations.
- $\Delta_\alpha = 0$: exact marginal deformations.
- $\Delta_\alpha < 0$: irrelevant deformations.

At a generic point on the Coulomb branch parameterized by λ_α , the low energy effective field theory is a $U(1)^r$ abelian gauge theory. and the corresponding three-fold $F(z, \lambda) = 0$ is smooth. If the three-fold is singular, we get interesting $\mathcal{N} = 2$ SCFT in the IR.

We would like to compare the spectrum of the UV theory defined by the hypersurface singularity f and the IR theory on the Coulomb branch. Since $\mathcal{N} = 2$ spectrum is determined by spectrum of the singularity and the RG flow is triggered by mini-versal deformation, the equivalent problem is how the singularity spectrum is changed under the deformation.

Interestingly, there is a remarkable theorem about how the singularity spectrum changes under deformation. This is called semicontinuity of spectrum as first conjectured by Arnold [13]. The theorem relevant for us is proven by Steenbrink [17]. Let's explain some details about this theorem.

Assume that a polynomial f has singular points x_1, x_2, \dots, x_n , we define its spectrum as the union of the spectra of each singular point x_i :

$$sp(f) = \sum_{i=1}^n sp(f(x_i)). \quad (4.1)$$

We define the spectrum in the subset B of real line as the number of spectrum numbers in the region B :

$$|sp_B(f)| = sp(f) \cap B, \quad (4.2)$$

B is called a semicontinuity domain if

$$|sp_B(f_{UV})| \geq |sp_B(f_{IR})|. \quad (4.3)$$

Here f_{UV} is the isolated singularity we start with, and f_{IR} is the polynomial at any point on the Coulomb branch. Steenbrink [17] proves the following theorem:

Theorem 1. Every half open interval $(t, t+1]$ is a semicontinuity domain of deformations of isolated hypersurface singularity.

In next subsections, we are going to study various interesting physical consequences of this theorem.

4.1 Consequences

4.2 Semicontinuity of $\mathcal{N} = 2$ spectrum

Let's denote the UV spectrum as $l_1 \leq l_2 \leq \dots \leq l_\mu$, and the IR spectrum as $l'_1 \leq l'_2 \leq \dots \leq l'_{\mu'}$. Using the theorem 1, we have following facts about the change of spectrum under RG flow:

- The first result is: $l'_1 \geq l_1, l'_2 \geq l_2, \dots$, etc.

Proofs. If $l'_1 < l_1$, then we choose a domain $B = (l'_1 - 1, l'_1]$, then $|sp_B(f_{UV})| = 0$ and $|sp_B(f_{IR})| = 1$, which violates the theorem 1. Similarly, one can prove $l'_2 \geq l_2$, etc.

We now translate the semicontinuity of the singularity spectrum into the spectrum of the corresponding $\mathcal{N} = 2$ theory. Using the formula (3.7), we find that: if we denote the UV spectrum as $\Delta_1 \geq \Delta_2 \geq \dots \geq \Delta_\mu$, and the IR spectrum as $\Delta'_1 \geq \Delta'_2 \geq \dots \geq \Delta'_{\mu'}$, we have

$$\Delta_1 \geq \Delta'_1, \Delta_2 \geq \Delta'_2, \dots \quad (4.4)$$

- The second consequence is that $\mu' \leq \mu$. If $\mu' = \mu$, then the spectrum would be the same. We leave the proof to interested reader. Since μ constant strata in the SW geometry is described by the marginal and irrelevant deformations [29], this result agrees with what is expected from $\mathcal{N} = 2$ superconformal algebra.

Example. Let's consider a simple example to illustrate the above theorem. Consider the flow from A_3 AD theory to A_2 AD theory. The spectrum of UV theory is $(\frac{4}{3}, 1, \frac{2}{3})$, and the IR spectrum is $(\frac{6}{5}, \frac{4}{5})$, so clearly the spectrum satisfy the semicontinuity property.

4.3 A theorem

For the theory defined by hypersurface isolated singularity, the central charge can be computed by the following formula [32]:

$$a = \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5r}{24}, \quad c = \frac{R(B)}{3} + \frac{r}{6}. \quad (4.5)$$

Here r is the rank of Coulomb branch (operator with scaling dimension bigger than one); $R(A)$ can be computed using the spectrum found from SW curve, and $R(B)$ is related to the number of A_1 singularity on Coulomb branch:

$$R(A) = \sum_{[u_i] > 1} ([u_i] - 1), \quad R(B) = \frac{\mu}{4(\sum q_i - 1)} = \frac{\mu \Delta_1}{4}. \quad (4.6)$$

Using theorem 1, we have

$$R(A)_{\text{UV}} \geq R(A)_{\text{IR}}, \quad R(B)_{\text{UV}} \geq R(B)_{\text{IR}}, \quad r_{\text{UV}} \geq r_{\text{IR}}; \quad (4.7)$$

and we have the equal sign for all three numbers if and only if the deformations are exact marginal or irrelevant deformation. So we have proven that $a_{\text{UV}} > a_{\text{IR}}$ for the flow triggered by turning on relevant deformations and Coulomb branch moduli.

4.4 Absence of dangerous irrelevant deformations

A dangerous irrelevant deformation is defined as a deformation which is irrelevant in the UV, but become relevant in the IR. In our setup, those deformations are defined as deformation with negative scaling dimension. Notice that from our definition, the operator with scaling dimension larger than two is regarded as **relevant** as one can turn on expectation value of these operators and flow to a different IR theory. Using theorem 1, a deformation with negative scaling dimension can never become relevant in the IR.

4.5 Necessary condition for the ending point of a RG flow

One would like to ask whether a theory A can appear as the ending of a RG flow starting from theory B . A necessary condition for this to happen is that the spectrum of A and B should satisfy theorem 1.

5 Conclusion

In this paper, we found the semicontinuity behavior of $\mathcal{N} = 2$ spectrum under RG flow on Coulomb branch for theory defined by isolated hypersurface singularity. The weaker statement that the scaling dimension of ordered spectrum is decreasing under RG flow can be easily checked for some RG flows of class \mathcal{S} theory [33, 34] using the result found in [27]. It would be interesting to check the stronger statement of semicontinuity and find out the semicontinuity domain for this class of theories. It would be also interesting to explore how spectrum changes under RG flow for general $\mathcal{N} = 2$ theory, and we expect that similar semicontinuity behavior is always true.

Notice that the Higgs branch spectrum do not have the monotonically decreasing property under RG flow on Coulomb branch. For example, one can flow from A_{2n} AD theory to A_{2n-1} AD theory on Coulomb branch. The UV theory has no Higgs branch, but the IR theory do have a Higgs branch.

The semicontinuity property is clearly applied to two dimensional $(2, 2)$ SCFT defined by Landau-Ginzburg model [35, 36]. For a two dimensional $(2, 2)$ theory defined by a hypersurface isolated singularity with C^* action $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$, the (c, c) ring is identified with the Jacobi algebra J_f [35], and one can define its spectrum as the q charges of coupling constants of the following deformation

$$\delta S = \int d^2\theta \lambda \mathcal{O} + \text{c.c.} \quad (5.1)$$

Here \mathcal{O} is a (c, c) ring element. We have the following set of rational numbers:

$$(q_1, \dots, q_\mu); \quad q_1 \geq q_2 \geq \dots \geq q_\mu. \quad (5.2)$$

These numbers are related to singularity spectrum as $q_i = 1 - l_i + l_1$. The RG flow is again related to the deformation of singularity, and we have the semi-continuity of $(2, 2)$ spectrum under RG flow using theorem 1. The physical consequences are similar, i.e. the proof of a theorem, the absence of dangerous irrelevant operator, etc.

Similar behavior about the change of spectrum is also found for RG flow of some four dimensional $\mathcal{N} = 1$ SCFTs. The relevant objects are of course the spectrum of chiral ring elements [37]. Consider $\mathcal{N} = 1$ SQCD in conformal window, the chiral spectrum is the mesons $M_i^j = Q_i \tilde{Q}_j$, baryons $B = Q^N$ and anti-baryons $\tilde{B} = \tilde{Q}^N$. The R charge of these operators are $R(M_i^j) = 2 \frac{N_f - N_c}{N_f}$, $R(B) = R(\tilde{B}) = \frac{N_c(N_f - N_c)}{N_f}$. We can turn on mass parameter to one of the flavor to flow from (N_c, N_f) theory to $(N_c, N_f - 1)$ theory, and the scaling dimensions of chiral operators all decrease under flow. It would be interesting to further study how the chiral spectrum is changed under RG flow for general $\mathcal{N} = 1$ theory, and we would like to come to this question in the near future.

Finally, it is interesting to explore how chiral spectrum changes under RG flow for supersymmetric theories in various dimensions. It is also interesting to look at the change of full spectrum under RG flow.

Acknowledgments

We would like to thank S.Gukov, Yifan Wang and Wenbin Yan for helpful discussions. The work of S.T Yau is supported by NSF grant DMS-1159412, NSF grant PHY- 0937443, and NSF grant DMS-0804454. The work of DX is supported by Center for Mathematical Sciences and Applications at Harvard University, and in part by the Fundamental Laws Initiative of the Center for the Fundamental Laws of Nature, Harvard University.

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